Enrollment No: ______________________ **Exam Seat No:** ______________________

C.U.SHAH UNIVERSITY

Summer Examination-2022

Subject Name: Theories of Ring and Field

Subject Code: 5SC03TRF1 **Branch: M.Sc. (Mathematics)**

Semester: 3 **Date: 26/04/2022** **Time: 02:30 To 05:30** **Marks: 70**

Instructions:

(1) Use of Programmable calculator and any other electronic instrument is prohibited.
(2) Instructions written on main answer book are strictly to be obeyed.
(3) Draw neat diagrams and figures (if necessary) at right places.
(4) Assume suitable data if needed.

SECTION – I

Q-1 **Attempt the Following questions** **(07)**

a. Define: Ring (01)

b. Define: Ideal (01)

c. Define: Characteristic of a ring. (01)

d. Write characteristic of Z_p where p is prime. (01)

e. Define: Prime Ideal. (01)

f. Every maximal ideal is prime ideal. True/False. (01)

g. Define: Euclidean ring. (01)

Q-2 **Attempt all questions** **(14)**

a. Prove that every field is an Euclidean ring. (05)

b. Prove that $R|I$ is an integral domain if and only if I is prime. (05)

c. Prove that every finite integral domain is field. (04)

OR

Q-2 **Attempt all questions** **(14)**

a. Prove that every Euclidean ring is principal ideal ring. (06)

b. Prove that for any two non-zero elements a and b in Euclidean ring R have L.C.M. in R. (04)

	c.	Prove that Z_p is field, where p is prime.	(04)
Q-3		**Attempt all questions**	**(14)**
	a.	If a is an element in a commutative ring R with unity then prove that the set $S = \{ra \mid r \in R\}$ is a principal ideal of R generated by a.	(05)
	b.	State and prove unique factorization theorem.	(05)
	c.	State and prove Gauss lemma.	(04)
Q-3		**Attempt all questions**	**(14)**
	a.	Let R be a Euclidean ring. Let $a, b \in R$ not both of which are zero. Then prove that a and b have a greatest common divisor d which can be express in the form of $d = \lambda a + \mu b$ where $\lambda, \mu \in R$.	(06)
	b.	Let R be Euclidean ring. Let a and b be two non-zero elements in R. If b is not unit in R then prove that $d(b) < d(ab)$.	(04)
	c.	Find all units of Gaussian integer.	(04)

SECTION – II

Q-4		**Attempt the Following questions**	**(07)**
	a.	Every principal ideal ring is Euclidean ring. True/False	(01)
	b.	Define: Degree of an algebraic element.	(01)
	c.	Define: Splitting field.	(01)
	d.	Define: Normal extension.	(01)
	e.	Define: Associates.	(01)
	f.	Define: Root of polynomial.	(01)
	g.	Is $f(x) = x^5 + 9x^4 + 12x^2 + 6$ irreducible over Q?	(01)
Q-5		**Attempt all questions**	**(14)**
	a.	Prove that $F[x]$ is a principal ideal ring.	(06)
	b.	If π is prime element in the Euclidean ring R and $\pi \mid ab$ where $a, b \in R$ then prove that π divides at least one a or b.	(05)
	C.	If $f(x) = 3x^4 + x^3 + 2x^2 + 1$ and $g(x) = x^2 + 4x + 2$ in $Z_5[x]$ then find quotient and remainder in $Z_5[x]$ when $f(x)$ is divided by $g(x)$	(03)

OR

Q-5		**Attempt all questions**	**(14)**
	a.	State and prove division algorithm for polynomials over field.	(06)
	b.	State and prove remainder theorem.	(04)
	c.	Prove that cyclotomic polynomial is irreducible over Q.	(04)

Q-6		**Attempt all questions**	**(14)**
	a.	State and prove Eisenstein criterion.	(05)
	b.	If $f(x)$ and $g(x)$ are primitive polynomials in $Z[x]$ then prove that $f(x)g(x)$ is primitive polynomial in $Z[x]$.	(05)
	c.	Let K be an extension field of F. Let $a \in K$ be algebraic over F. Then prove that any two minimal monic polynomials for a over F are equal.	(04)

OR

Q-6		**Attempt all Questions**	**(14)**
	a.	Prove that every finite extension K of a field F is algebraic.	(06)
	b.	Let K be an extension field of a field F. Then prove that $a \in K$ is algebraic over F if and only if $F(a)$ is a finite extension of F.	(06)
	c.	Show that the polynomial $x^2 + x + 4$ is irreducible over a field of integers modulo 11.	(02)